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14. ABSTRACT
The objective of this basic research project is to derive and interrogate mathematical models of the propagation of ultrashort pulses in mode locked lasers (particularly Ti:sapphire), the nonlinear propagation of light in certain optical waveguides and photonic lattices, and the nonlinear propagation of light in "chi(2)" media. The PI and his research team will study the nonlinear partial differential equations that arise from the full nonlinear Maxwell equations under various asymptotic limits. He will also employ elements of soliton theory as well as the inverse spectral transform. In addition he will pursue numerical approaches. The AF would be able to exploit mode locked lasers to speed up the calculation of cross-correlations between reference and backscattered signals in radars and would be able to exploit the optical switching promised by lattices and chi(2) materials to field circuitless computers.

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Nonlinear Wave Propagation
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OBJECTIVES

To carry out fundamental and wide ranging research investigations involving the nonlinear wave propagation which arise in physically significant systems with emphasis on nonlinear optics. The modeling and computational studies of wave phenomena in nonlinear optics include ultrashort pulse dynamics in mode-locked lasers, localized modes in waveguide arrays and photonic lattices, novel phenomena in quadratic nonlinear media and the dynamics of dispersive shock waves.

STATUS OF EFFORT

The PI's research program in nonlinear wave propagation is broad based and very active. There have been a number of important research contributions carried out as part of the effort funded by the Air Force. During the period 15 March 2006 – 30 November 2008, eight papers were published in refereed journals, two book chapters were published, two refereed conference proceedings were published and nineteen invited lectures were given. The key results and research directions are described below in the section on accomplishments/new findings. Full details can be found in our research papers which are also listed at the end of this report.

Research investigations carried out by the PI and colleagues included the following. The modes, dynamics and properties of mode-locked lasers which are used to create ultrashort pulses were analyzed. Titanium:sapphire (Ti:sapphire or Ti:s) lasers are often used to produce ultrashort pulses on the order of a few femtoseconds. There are other mode-locked lasers which produce ultrashort pulses, such as Sr-Forsterite, fiber lasers, and Chromium-doped lasers. Ti:s lasers are known to have outstanding characteristics. These mode-locked lasers can be used to generate a regularly spaced train of ultrashort pulses separated by one cavity round-trip time. A typical mode-locked laser system consists of a Ti:sapphire crystal which exhibits a nonlinear Kerr response and has a large normal group-velocity dispersion (GVD). This requires a set of prisms and/or mirrors specially designed to have large anomalous GVD in order to compensate for the normal GVD of the crystal. Recent experiments conducted at the University of Colorado, in collaboration with our group, demonstrated that such lasers can be approximated by dispersion-managed systems and the intra-cavity pulse was found to be described by a dispersion-managed soliton. Improved mathematical models reflecting gain and loss mechanisms are being developed; these models contain gain and filtering terms saturated by energy and a loss term saturated by power. The new models describes the mode-locking and dynamics of solitons.

A characteristic of short pulse lasers is the carrier-envelope phase (CEP) slip which is the change in phase between carrier and envelope from pulse to pulse in the pulse train. This is the phase slip that the intra-cavity pulse accumulates over one cavity round-trip before being emitted from the output coupler. The intra-cavity slip is affected by the nonlinearity and dispersion of the cavity. Control of the phase slip allows researchers to stabilize trains of ultrashort pulses which are useful for applications. Improved understanding of the phase slip and noise characteristics will help experimentalists improve the characterisitcs/stabilization of the pulse train.

Investigations of pulse propagation in photonic lattices were carried out. There has also

been important experimental research on discrete optical wave-guides and the propagation of their nonlinear modes. Experimentalists have been able to construct one and two dimensional lattices by interfering laser beams. This all-optical technique is a significant advantage over prior methods in which the background lattice structure was developed by mechanical fabrication.

Early experimental observation of one-dimensional nonlinear lattice modes in corrugated optical waveguide arrays demonstrated that at sufficiently high power, a laser beam could be self-trapped inside the waveguide. This demonstrated the formation of a lattice or discrete soliton. Importantly, such waveguides can be constructed on extremely small scales and, as mentioned above, recently have been constructed by all-opitcal means. In turn, such nonlinear waves in waveguide arrays have attracted special attention due to their realizability. Lattice nonlinear Schrödinger equations provide excellent models. We developed asymptotic and computational methods which describe observed phenomena and found localized pulse solutions to two-dimensional optical lattices with both regular and irregular lattice backgrounds. Irregular lattice backgrounds include vacancy defects, edge dislocations and quasi-crystal structures.

Research involving quadratic, or so-called $\chi^{(2)}$, nonlinear optical materials has led to a novel asymptotic systems of equations. Detailed calculations indicate that, in certain parameter regimes, there are stable localized pulse solutions. In other cases, the equations have unstable and singular solutions. The possibility of such singular solutions indicates situations when extreme damage to the underlying optical crystal is possible.

Dispersive shock waves (DSW's) were investigated. With experimentalists at the University of Colorado, dispersive shock "blast" waves and interactions were studied. The relevant analytical approximations and theory for DSW's were developed and results were compared with experiment and computation; key differences between DSW and viscous shock waves were also described. DSW's arise in nonlinear optics and many other areas of physics.

ACCOMPLISHMENTS/NEW FINDINGS

Dynamics of ultra-short laser pulses and frequency combs

Research breakthroughs over the past few years with mode-locked lasers, such as Ti:sapphire lasers, have enabled scientists to generate regularly spaced trains of ultrashort pulses, which are separated by one cavity round-trip time. Fig. ?? below shows a schematic of a mode-locked Ti:sapphire laser and the emitted pulse train. Typical values for a Ti:sapphire mode-locked laser are pulse width: $\tau = 10$ fs $= 10^{-14}$ sec and repetition time: $T_{\rm rep} = 10$ ns $= 10^{-8}$ sec.

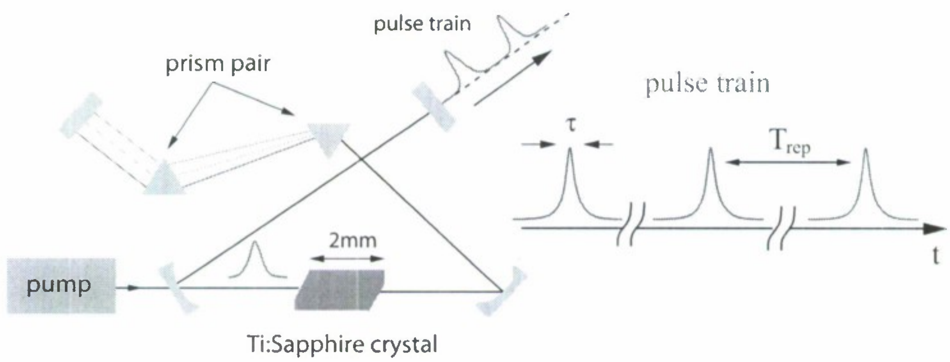

Figure 1: Ti:sapphire laser (left) and the emitted pulse train (right).

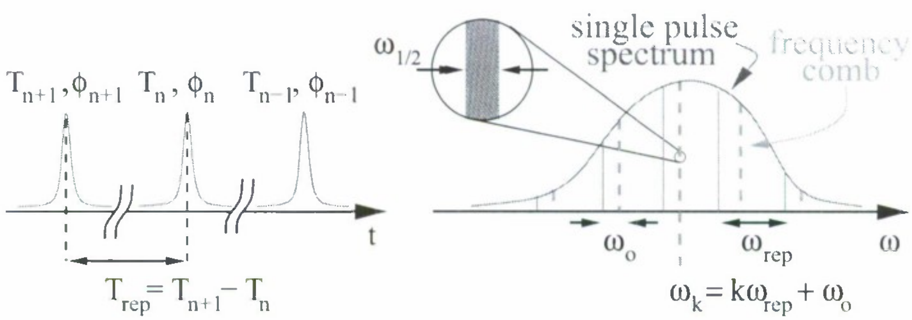

Figure 2: Schematic of a pulse train (left) and its spectrum -frequency comb (right).

Associated with the spectrum of the pulse train is a frequency comb, whose frequencies are separated by the laser's repetition frequency $f_{\rm rep} = \frac{1}{T_{\rm rep}} = 100$ MHz; $\omega_{\rm rep} = 2\pi f_{\rm rep}$; see

Fig. ?? above. In this figure T_n, ϕ_n are the center time and phase of the nth pulse. In the absence of noise, the pulse's spectrum determines the bandwidth, while the repetition-time, $T_{rep} = T_{n+1} - T_n$, and overall phase slip, $\Delta\phi = \phi_{n+1} - \phi_n$, determine the comb function in frequency space. The frequency of the k'th comb line (enumerated around the center frequency) is $\omega_k = k\omega_{\text{rep}} + \omega_o$, where $\omega_{\text{rep}} = 2\pi/T_{rep}$ and $\omega_o = \Delta\phi\omega_{\text{rep}}/2\pi$ are the repetition and offset frequencies which are depicted in Fig. ??. The linewidth ($\omega_{1/2}$ in the inset) is the FWHM (full width half maximum) of the comb function around each comb frequency. In the deterministic case, for a large number of pulses $N >> 1$, the linewidth can be estimated to be $\omega_{1/2} = O(\frac{1}{NT_{\text{rep}}})$. Additional noise leads to jitter in the center time, T_n, and phase, ϕ_n, which in turn broadens the FWHM of the comb lines.

As discussed above, mode-locked lasers such as Ti:sapphire (Ti:s) laser systems (see Fig. ??) can generate trains of optical pulses, whose spectrum consists of frequency comb lines (see Fig. ??). Important progress in the development of extremely stable optical oscillators has been made possible by the use of controlled femtosecond frequency combs. Extremely stable frequency combs have been generated by Ti:s laser systems, but others such as Sr:Forsterite lasers are also being studied intensively. We have been working with faculty in the Department of Physics at the University of Colorado on this research.

In our research we have been studying a distributed dispersion-managed, power energy saturation (PES) model system. For a pulse with amplitude $u(z,t)$, power $P(z,t) = |u|^2$, and energy $E(z) = \int_{-\infty}^{+\infty} |u|^2 \, dt$, which is propagating in the z direction, our normalized equation takes the form

$$i\frac{\partial u}{\partial z} + \frac{d(z)}{2}\frac{\partial^2 u}{\partial t^2} + n(z)|u|^2 u = \frac{ig}{1+E/E_{\text{sat}}}u + \frac{i\tau}{1+E/E_{\text{sat}}}u_{tt} - \frac{il}{1+P/P_{\text{sat}}}u \tag{1}$$

where the constant parameters g, τ, l, E_{sat}, P_{sat} are positive. The first term on the right hand side represents saturable gain, the second is nonlinear filtering ($\tau \neq 0$) and the third is saturable loss. This model generalizes the well-known master laser equation. An important

observation is that when the loss term is approximated in the weakly-nonlinear regime by a first order Taylor polynomial we obtain the master laser equation. Hence, the master laser equation is included in the power saturated model as a first order approximation.

The dimensionless governing equation is obtained by transforming to the following variables:

$$z' = z/z_*, t' = t/t_*, u = E/\sqrt{P_*}$$

where E is the envelope of the electromagnetic field t_*, P_* are the characteristic time (proportional to pulse width) and power respectively. We take $z_* = 1/(P_*\gamma_0)$, where γ_0 (in units 1/MW-mm) is the nonlinear coefficient in the laser crystal, g, τ and l have all been scaled by z_* and the normalized dispersion is given by

$$d(z) = -k''(z)/k''_*, \quad k''_* = t_*^2/z_*,$$

where k'' is the GVD (in units fs^2/mm). In normalized units we find

$$d(z) = \langle d \rangle + \frac{\Delta(z)}{l_c},$$

with $\langle d \rangle$ being the average dispersion (net GVD), Δ the deviation from the average GVD, and l_c the normalized laser-design map length. Usually one considers a two step dispersion map where Δ_j, $j = 1, 2$ are taken to be constant in the mirror+prism components ($j = 1$) and crystal ($j = 2$). The map length over which the anomalous dispersion occurs is θl_c where $0 < \theta < 1$; typically we take $\theta = 1/2$ in Ti:sapphire laser applications. We also introduce the map strength parameter s, which is proportional to the area under the dispersion map, $s = \frac{1}{4}[\theta\,\Delta_1 - (1-\theta)\,\Delta_2]$. In addition to dispersion-management, we also have nonlinear-management in this laser model . Here nonlinear-management means that $n = 1$ (transformed from γ_0 in dimensional variables) inside the Ti:sapphire crystal and $n = 0$ outside the crystal; that is, we assume linear propagation inside the prisms and mirrors. In Fig. ?? below a schematic diagram of the normalized dispersion- managed configuration associated with a mode-locked Ti:sapphire laser system is shown.

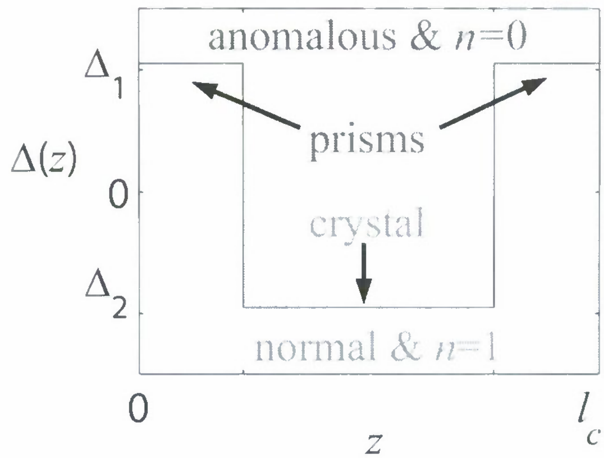

Figure 3: Figure shows a typical dispersion-managed configuration.

In our earlier research investigations in fiber optics we derived, based on the asymptotic procedure of multiple scales, a nonlinear integro-differential equation (not given here due to space considerations) which governs the dynamics of dispersion-managed pulse propagation. This governing equation is referred to as the dispersion-managed nonlinear Schrödinger (DMNLS) equation. When there is no gain or loss in the system, for strongly dispersion-managed systems, the DMNLS equation plays the role of the "pure" NLS equation–which is the relevant averaged equation when there is either small or no dispersion-management. When gain and loss are included as in the PES equation we have a modification of the "pure" DMNLS equation.

With or without dispersion-management the PES equation naturally describes the locking and evolution of pulses in mode-locked lasers that are operating in the soliton regime. To describe our research in more detail, we fix typical values of the parameters, vary the gain parameter g and the map strength s. When $g < g^*(s)$ no localized solution is obtained; i.e. in this case the effect of loss is stronger than a critical gain value and the evolution of a Gaussian profile decays to the trivial solution. Conversely, when $g > g^*(s)$, there exists a single localized solution, $u = U_0(t)\exp(i\mu z)$ where μ, called the propagation constant,

is uniquely determined given the specific values of the other parameters. The localized solutions of the modified DMNLS equation with gain-loss in the different regimes are shown in Fig. ??; in this figure we see that for given map strengths the amplitude of the pulses

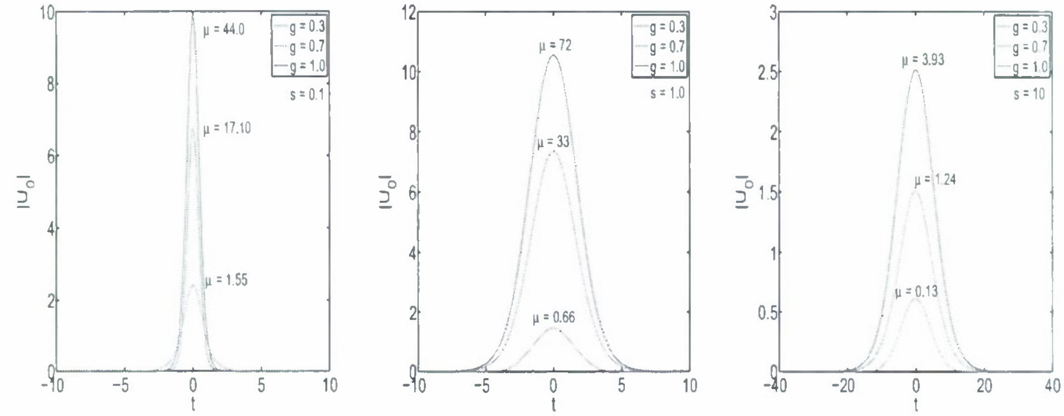

Figure 4: Solitons of the DMNLS equation with gain-loss corresponding to different map strengths and gain parameters. Notice that these solitons occur only for a specific value of propagation constant, μ, in contrast to the unperturbed case where solutions exist for all $\mu > 0$, thus providing the mode-locking mechanism.

increases with g. In addition, we see that the pulses become broader as the map strength, s, increases. Blow-up is not obtained, even under extreme gain. As mentioned earlier, the influence of gain-loss on pulses in this model is to create the mode-locking mechanism. The resulting modes also correspond closely to the modes of the unperturbed equation with the same propagation constant μ.

Interestingly it is known that only for a narrow range of parameters the master laser equation (when the loss term is taken to be the first two terms of the Taylor expansion of the last, power saturated term, in the PES equation) exhibits stable soliton solutions with mode-locking evolution. Otherwise the solitons are found to be unstable; either dispersing to radiation or evolving into nonlocalized quasi-periodic states. In addition, for diffrerent parameters, the amplitude can also grow rapidly under evolution. Thus, the basic master laser equation captures some qualitative aspects of pulse propagation in a laser cavity; however,

since there is only a small range of the parameter space for which stable mode-locked soliton pulses exist, it does not reflect the wide ranges of operating conditions where mode-locking occurs.

The PES and the master laser equation have parameters which are obtained from experiments. When there is insufficient gain in Eq. (??), both models show that pulses dissipate to zero. Remarkably, a distinguishing feature of the PES model is that even under large gain, pulses do not blow-up nor do they exhibit instabilities. Indeed, when the gain is greater than some threshold value $g = g^*$, during the evolution the pulse readjusts itself as it mode locks into a stable localized mode, or soliton solution, which we refer to as a soliton wave attractor (SWA). Furthermore, all pulses that evolve into SWAs can be obtained independently using a mode finding algorithm which we developed in our earlier research supported by AFOSR.

Power saturation models also arise in other problems in nonlinear optics and are important in the underlying theory. For example, power saturation models are important in the study of the dynamics of localized lattice modes (solitons, vortices, etc) propagating in photorefractive nonlinear crystals. If the nonlinear term in these equations was simply a cubic nonlinearity, without saturation, two dimensional fundamental lattice solitons would be vulnerable to blow up singularity formation, which is not observed. Thus saturable terms are crucial in these problems.

Carrier-envelope phase slip

As mentioned above, mode-locked lasers can generate a regularly spaced train of ultrashort pulses separated by one cavity round-trip time. The phase slip is the change of the phase offset between carrier and envelope from pulse to pulse in the pulse train which accumulates over one cavity round-trip, before being emitted from the output coupler. Fig. ?? depicts the physical origin of the carrier envelope phase (CEP) shift. The intra-cavity slip is induced by the nonlinearity and dispersion of the cavity. The phase offset ω_o in the frequency comb is proportional to the carrier envelop phase $\omega_o = \Delta\phi\omega_{\text{rep}}/2\pi$ where the

repetition frequency is $\omega_{\text{rep}} = 2\pi/T_{rep}$ and the carrier envelop phase is given by $\Delta\phi_{CE} \equiv \Delta\phi$.

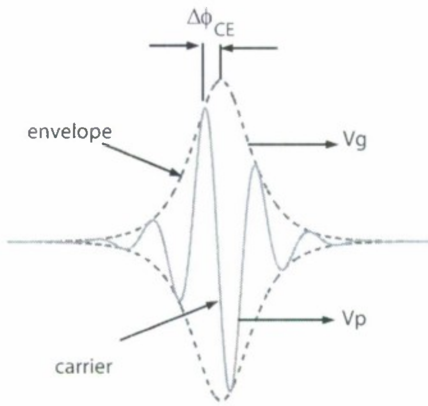

Figure 5: The carrier-envelope phase, $\Delta\phi_{CE} \equiv \Delta\phi$, changes during propagation, because the envelope propagates at the group velocity while the carrier wave propagates at the phase velocity.

A typical Ti:s laser, such as the one depicted in Fig. ??, consists of a Ti:sapphire crystal that has a nonlinear Kerr response as well as large normal group-velocity dispersion (GVD), and a set of prisms and mirrors specially designed to have large anomalous GVD. The pump laser excites the Ti:Sapphire crystal, causing it to lase and the pulse undergoes large changes inside the Ti:sapphire cavity . The combined contributions to the phase slip depends on the nonlinear phase and nonlinear dispersion in the cavity. The crystal induces a nonlinear effect as well as large *normal* GVD; the mirror and prisms induce large *anomalous* GVD that nearly balances the normal GVD of crystal with a small net-positive GVD (average dispersion) over one round-trip. The pulse bounces between the mirrors and output coupler and is "sampled" every round-trip at the output coupler (which transmits only 6% of the energy). When the laser is mode-locked a regularly-spaced ultrashort pulse train is emitted from the cavity. In our earlier work we employed a perturbed NLS equation without gain/loss; i.e. we took $g = \tau = l = 0$ in Eq. (??), but we added a small nonlinear term on the RHS of proportional to $n(z)(|u|^2u)_t$ and a higher order (third order) linear dispersive term (see below). The

nonlinear term on the right-hand side, often called the "shock" term, corresponds to small nonlinear dispersion arising from the Kerr effect. The shock term is particularly important for shorter pulses and it induces a nonlinear change in the phase slip, which is known to depend on pulse energy (pump power).

We found the nonlinear slip, i.e., the slip induced by nonlinear phase and nonlinear dispersion effects, to be well-approximated by

$$\delta_{\mathrm{NL}} \approx 3\bar{k}'' l_c / \tau_0^2 s,$$

where $\bar{k}''$ is average-cavity GVD, τ_0 is the pulse width, l_c is the optical cavity length and s is the map strength. This result shows that the phase slip that is induced by nonlinear-dispersion vanishes with stronger dispersion-management, which is consistent with the insensitivity of the slip to pulse energy with strong dispersion-management. An additional effect which impacts the phase slip is third-order dispersion (TOD), which can be modeled by adding $[i\bar{k}'''/(6\gamma_0 P_* \tau_0^3)]u_{ttt}$ (normalized) to the right-hand side of Eq. (?? (with $g = \tau = l = 0$), where $\bar{k}'''$ is the average TOD coefficient, i.e., the net TOD per round-trip of the cavity and P_* is the characteristic power.

Thus the laser cavity was modeled by a dispersion and nonlinearity managed nonlinear Schrödinger equation (perturbed NLS), that takes nonlinear phase (self-phase modulation) and small nonlinear dispersion into account. In our work we developed a detailed asymptotic theory which gave analytic results that described the carrier-envelope phase shift in this dispersion-managed NLS equation. Control of the carrier envelope phase shift is important in applications. It is a key aspect to obtain highly stable optical oscillators.

Comparison of theory with experiments

Motivated by the question of whether or not the mode-locked pulses observed in the laboratory were actually solitons, a number of experiments were performed in the Dept. of Physics at the University of Colorado. It was found that the "pure" dispersion-managed theory, without gain/loss terms (i.e. in this case $g = \tau = l = 0$ in Eq. (??)) agrees with

experiments remarkably well. Moreover, in recent work accounting for $g, \tau, l \neq 0$ we found that when the system mode locks, the solitons are well approximated by pure dispersion-managed solitons. This further supports the comparisons of theory and experiment and the carrier envelope phase slip calculation described above. In Fig. ?? the full width half maximum (FWHM) is plotted vs. the average pulse energy for various values of the net group delay dispersion (GDD). The dashed lines are the theoretical values. This experimental research also shows that dispersion-management soliton concepts are broadly applicable.

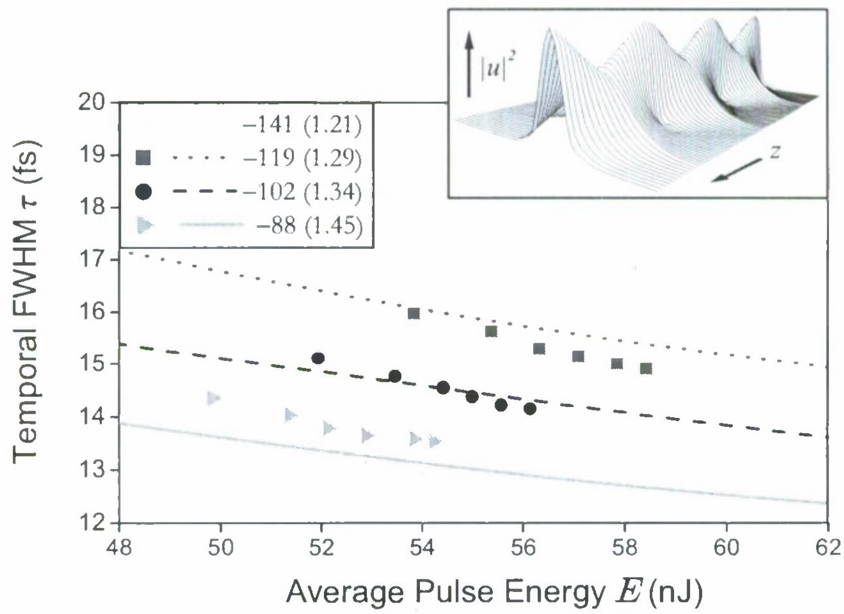

Figure 6: Fundamental pulse parameters in a mode-locked Titanium:sapphire laser. The points are the measured temporal FWHM at four values of the average cavity GDD. The curves are the solutions of the Dispersion Managed NLS equation. The legend states the GDD values in fs^2. The errors for the GDD are approximately 1% and the errors for the τ values are negligible on the scale shown. Inset shows the breathing dynamics of a dispersion-managed soliton $|u|^2$ as it propagates along z.

Noise induced linewidth in frequency combs

We have discussed earlier that mode-locked Ti:s lasers generate trains of optical pulses, whose spectrum consists of frequency comb lines. These combs are represented by evenly-spaced frequencies and an offset frequency that is proportional to the carrier envelope phase (see Fig. ??).

Random physical effects can induce a linewidth, or uncertainty, in a comb line frequency. Some random effects can be minimized but, as far back as 1958, Schawlow and Townes discovered that the linewidth, or monochromaticity, of a single-mode continuous-wave (cw) laser is fundamentally limited by the random process of amplified spontaneous emission (ASE) in the lasing medium. We studied the limits of frequency combs associated with mode-locked lasers in the presence of ASE noise and obtained a scaling law result. In particular, we analyzed the frequency combs generated by trains of pulses emitted from mode-locked lasers when the center-time and phase of the pulses undergo noise-induced random walk, which in turn broadens the comb lines. Detailed asymptotic analysis and computation of the ensemble-averaged spectrum revealed a time-frequency duality. In particular, the increase of the standard deviation of the center-time with pulse number, and the increase of the linewidth with frequency, are closely related. More precisely, the standard deviation of the center-time jitter of the n'th pulse is found to scale as $n^{p/2}$, where p is a jitter-exponent, while the linewidth of the k'th comb line scales as $k^{2/p}$. The linear-dispersionless system ($p = 1$) and pure nonlinear soliton ($p = 3$) dynamics in lasers are found as special cases of this time-frequency duality relation.

Although this result was derived for mode-locked lasers, the general nature of the result indicates that it can be applied to the stochastic dynamics associated with other frequency combs.

Nonlinear Optics in Waveguide arrays and photonic lattices

Nonlinear light wave propagation in photonic lattices, or periodic optical waveguides, is an active and interesting area of research. This is due, in part, to the realization that photonic lattices can be constructed on extremely small scales, typically a few microns in size. They allow the possibility of manipulation and navigation of lightwaves in small regions. Localized nonlinear optical pulses which occur on one and two dimensional backgrounds have been investigated. These backgrounds can be either fabricated mechanically such as those comprised of AlGaAs materials or all-optically using photo-refractive materials where the photonic structures are constructed via interference of two or more plane waves.

In two-dimensional photonic lattice applications, a nonlinear Schrödinger equation with an external potential, derivable from Maxwell's equations, is the governing equation. The equation in normalized form is

$$i\frac{\partial u}{\partial z} + \Delta u - V(x,y)u + |u|^2 u = 0 \tag{2}$$

Here $u(x,y,z)$ corresponds to the slowly-varying complex amplitude of the electric field in the plane that is propagating along the z direction, $\Delta u \equiv u_{xx}+u_{yy}$ corresponds to diffraction, $V(x,y)$ is an external optical potential, and the cubic term in u originates with the nonlinear-Kerr change of the refractive index for a cubic or $\chi^{(3)}$ nonlinear index. The optical potential corresponds to a change in the linear refractive index of the medium, which can be achieved either by etching or with an all-optical inductance technique. We note that the potential assumed in this study is uniform along the propagation direction z, although in principle, non-uniform potentials in z could also be considered in a manner analogous to diffraction or dispersion-management.

Recently we have begun a collaboration with faculty in the Electrical Engineering Department at the University of Colorado. They have constructed defect, dislocated photonic lattice systems and quasi-crystal potentials in the laboratory. By a defect we mean that only

one or two lattice sites are affected by either removing a small number of sites or enhancing the amplitude of these sites. A dislocated photonic lattice system has, for example, a line of sites merging with another. A quasi-crystal structure has long range order but is not periodic. The general quasi-crystal potential we study is given by

$$V(x,y) = \frac{V_0}{N^2} \left| \sum_{0}^{N-1} e^{ik[\cos(2\pi n/N)x + \sin(2\pi n/N)y]} \right|^2$$

which corresponds to the diffraction of plane waves that emanate from N equally spaced points on a circle. When $N = 2, 3, 4, 6$ regular periodic lattice backgrounds result. The simplest nontrivial (i.e nonperiodic) case is $N = 5$ which is often referred to as a Penrose potential—this is named after R. Penrose who discussed such functions in the mathematical context of tiling the plane. We are currently studying higher order and more general cases. Defect, dislocation and quasi-crystal structures are often seen in nature. Typical cases are given in Fig ?? below.

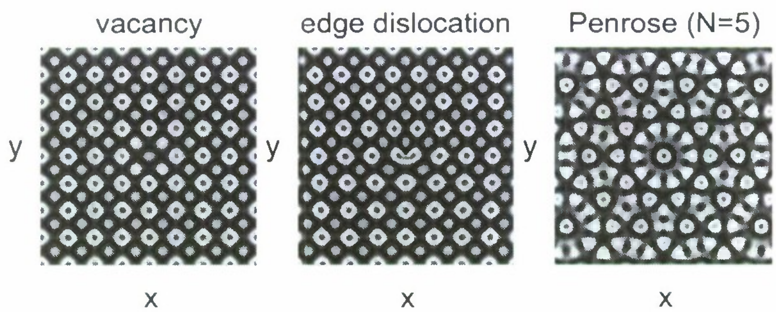

Figure 7: vacancy defect, left, dislocation, middle, quasi-crystal order $N = 5$ -Penrose structure-right

In our research we constructed, by using our recently developed computational methods, nonlinear localized pulses which are associated with complex defect photonic lattices such as the ones in Fig ??. To date most investigators have considered regular lattices. Photonic lattice systems and waveguide arrays, which we often approximate as discrete systems, are natural components of optical design. Since they can be fabricated in small physical envi-

ronments they allow the possibility of controlling and tuning light waves in confined regions and to navigate light in one and two dimensional networks.

Dynamics and pulse propagation in quadratic nonlinear optical media

In many applications the leading nonlinear polarization effect in an optical material is quadratic; they are usually referred to as $\chi^{(2)}$ materials. We have found that in multidimensional nonresonant $\chi^{(2)}$ materials, the nonlinear equation governing the slowly varying envelope of quasi-monochromatic wave trains is a coupled nonlinear system involving both the optical field and mean terms. They are a generalization of the classical nonlinear Schrödinger equation. We call these equations NLSM systems (M stands for the mean contribution). In water waves similar scalar systems were derived in 1969 by Benney and Roskes. A few years later, a special case of this system was found to be integrable. The latter system is freqnently referred to as the Davey-Stewartson (DS) system.

We have derived both scalar and a vector NLSM systems directly from Maxwell's equations. The vector NLSM systems generalize the well known 1+1 vector NLS equations to multidimensions. Such vector multidimensional systems are new in mathematical physics; they have not yet been derived in other physical systems.

In our research we found that the $\chi^{(2)}$ optical system can exhibit wave collapse corresponding to a suitable ranges of parameter and initial data. This indicates that intense optical pulses can occur in these systems. We believe that experimentalists will be able to observe this phenomena since the analagous situation was recently observed in cubic nonlinear media. These theoretical results indicate that in this range of parameters researchers must be careful in their experiments to prevent damaging the underlying optical crystal.

The NLSM system of equations possesses nonlocal-nonlinear coupling between a field that is associated with the first harmonic, with a "cascaded" effect from the second harmonic, and a static field that is associated with the mean term (i.e., the zero'th harmonic). The system that we analyzed in detail can be written in the following non-dimensional form,

$$iu_z + \frac{1}{2}\Delta u + |u|^2 u - \rho u \phi_x = 0 \ , \tag{3a}$$

$$\phi_{xx} + \nu \phi_{yy} = (|u|^2)_x \ , \tag{3b}$$

where $u(x, y, z)$ corresponds to the field associated with the first-harmonic, $\phi(x, y, z)$ corresponds to the mean field, and ν and ρ are real constants that depend on the physical parameters.

It can be proven that the above system can admit wave collapse of localized waves when $\nu > 0$. We remark that the initial conditions are $u(x, y, 0) = u_0(x, y)$, $\phi(x, y, 0) = \phi_0(x, y)$, such that equation (??) is satisfied at $z = 0$, i.e., $\phi_{0,xx} + \nu\phi_{0,yy} = (|u_0|^2)_x$.

We also note that our numerical solutions indicate that as wave collapse occurs, the solution tends to the steady state mode found from the above NLSM system (??). The steady solution is obtained by assuming a solution of the form $u(x, y, z) = F(x, y)e^{i\lambda z}$ and $\phi(x, y, z) = G(x, y)$, where F and G are real functions and λ is a positive real number. Substituting this ansatz into equations (??) gives

$$-\lambda F + \frac{1}{2}\Delta F + F^3 - \rho F G_x = 0 \ , \tag{4a}$$

$$G_{xx} + \nu G_{yy} = (F^2)_x \ . \tag{4b}$$

We plan to investigate whether localized optical pulses can be obtained when we have an underlying optical lattice. We are also investigating the possibility of light navigation and optical switching.

Dispersive Shock Waves

Shock waves in compressible fluids is a classically important field in applied mathematics and physics, whose origins date back to the work of Riemann. Such shock waves, which we refer to as classical or viscous shock waves (VSW's), are characterized by a localized steep gradient in fluid properties across the shock front. Without viscosity one has a mathematical discontinuity; when viscosity is added to the equations, the discontinuity is "regularized"

and the solution is smooth. An equation that models classical shock wave phenomena is the Burgers equation

$$u_t + uu_x = \nu u_{xx} \tag{5}$$

If $\nu = 0$, we have the inviscid Burgers equation which admits wave breaking. When the underlying characteristics cross a discontinuous solution, i.e. a shock wave, is introduced which satisfies the Rankine-Hugoniot jump conditions which, in turn, determines the shock speed. Analysis of Burgers equation shows that there is a smooth solution given by

$$u = \frac{1}{2} - \frac{1}{2}\tanh\{\frac{1}{4\nu}(x - \frac{1}{2}t)\}$$

which tends to the shock solution as $\nu \to 0$. Thus the mathematical discontinuity is regularized when viscosity ν is introduced.

Another type of shock wave is a so-called dispersive shock wave (DSW). Early observations of DSW's were ion-acoustic waves in plasma physics. Subsequently, Gurevich and Pitaevskii studied the small dispersion limit of the Korteweg-deVries (KdV) equation. They obtained an analytical representation of a DSW. As opposed to a localized shock as in the viscous problem, the description of a DSW is one with a sharp front with an expanding, rapidly oscillating rear tail. The Korteweg-deVries (KdV) equation with small dispersion is given by

$$u_t + uu_x = \epsilon^2 u_{xxx} \tag{6}$$

where $0 < \epsilon << 1$ regularizes the discontinuity that otherwise would be present. The mathematical technique used to analyze DSW's relies on wave averaging, often referred to as Whitham theory. Whitham theory is used to construct equations for the parameters associated with slowly varying wavetrains; it provides an analytical basis for DSW dynamics. For KdV the Whitham equations can be transformed into Riemann-invariant form, which can be analyzed in detail. One finds that there are two speeds associated with a DSW: one is

the speed associated with the frontal wave which is a soliton (located at x_s in the figure), and the other speed corresponds to the group velocity of near linear trailing waves on the rear end (depicted by x_l in the figure) of the DSW. The picture and details are quite different from viscous shock waves which, for example, occurs in Burgers equation; cf. the leftmost Fig. ?? which depicts a typical DSW associated with the KdV equation and a classical or viscous shock wave (located at x_c in the figure) associated with the Burgers equation. Interestingly, the structure of the KdV DSW is strikingly similar to the original plasma observations.

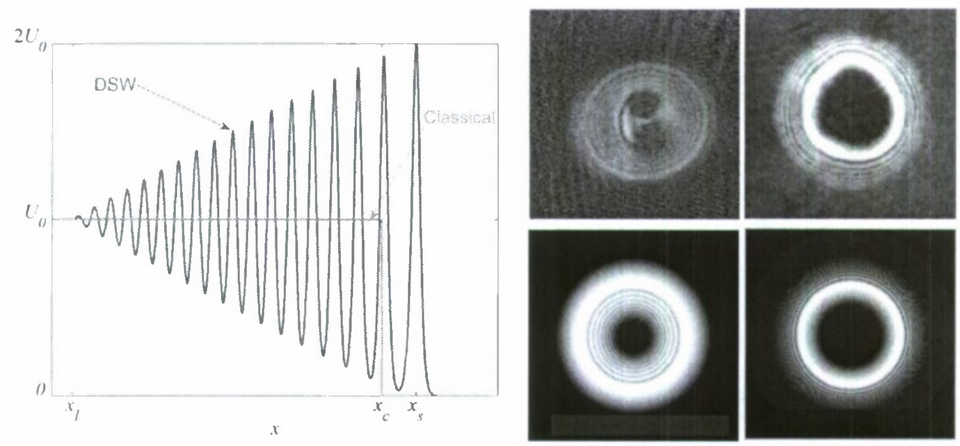

Figure 8: Left figure: typical DSW satisfying the KdV eq. (??) and a classical shock wave satisfying Burgers eq. (??); middle figure: "in-trap" blast wave; right figure: "out-of-trap" blast wave. Numerical simulations are given below those of the corresponding experiment.

Recent experiments in Bose-Einstein condensates (BEC) and nonlinear optics have enhanced interest in DSW's. The BEC experiments, originally performed in the Physics Department at the University of Colorado, motivated our studies. Shock waves emanating from a blast or explosion are well known in the study of viscous shock waves. Interestingly a similar situation occurs in the context of DSW's. The dispersive blast wave experiment and computational results are shown Fig. ?? – middle and right figures. Recent experiments in nonlinear optics carried out in the laboratory of J. Fleischer at Princeton University have also observed similar blast waves and other interesting DSW phenomena.

The governing equations we studied are a defocusing NLS equation with an additional external potential. In BEC this equation is usually called the Gross-Pitaevski equation. Importantly, a similar equation occurs in nonlinear optics. The equation is given in normalized form as

$$i\epsilon\Psi_t + \frac{\epsilon^2}{2}\nabla^2\Psi - V(r,z,t)\Psi - |\Psi|^2\Psi = 0 \tag{7}$$

where $V(r,z)$ contains details describing the cylindrically symmetric potential trap and additional laser terms, Ψ is the wavefunction and ϵ^2 is a small parameter related to the condensate. Two different configurations were studied: the so-called "in-trap" and "out-of-trap" cases. The two cases correspond to the turning on of a tightly focused laser beam when the BEC was in either a trapped or an expansion (the latter is called nontrapped) configuration. The laser beam creates a dispersive blast wave in the radial direction.

Numerical simulations were found to agree extremely well with the experiments. In Fig. ?? (middle and right figures), experimental and numerical results are shown for the two BEC configurations; the numerical image is a contour plot of $\int |\Psi|^2 dz$ (darker is less dense). Our initial analytical studies were carried out on the 1-D semi-classical NLS equation without the external potential since the experiments indicated that after some propagation time the DSW's were approximately 1-D in character and weakly influenced by the potential.

As indicated earlier, an analytical approach to DSW's uses Whitham averaging theory. Whitham analysis associated with NLS equations in various contexts have been investigated by a number of authors and in order to effectively compare the analysis with BEC experiment and simulations, by ourselves. Whitham averaging over the first four conservation laws of NLS, using a 1-phase traveling wave solution, leads to suitable parameters satisfying a system of hyperbolic PDE's which can be written in Riemann-invariant form. Solving the Riemann-invariant system corresponding to step initial data, yields a description of an NLS DSW. The NLS DSW is a slowly modulated wave train which varies from a large trailing wave which is well approximated by a moving dark (gray) soliton, to a nearly linear wavetrain at

the front moving with its group velocity; like KdV the NLS DSW has two speeds. The 1-D NLS theory was applied to both of the multi-dimensional blast wave cases. The analytical results were very good; however due to radial and potential effects there is a difference in phase and to a lessor degree in amplitude.

While interactions of viscous shock waves are well known, the situation associated with DSW's is still at an early stage. We have made some progress in our research, but more work still needs to be done in order to develop a broad and detailed understanding. We are investigating DSW interactions in physically interesting systems by employing both Whitham methods and asymptotic analysis applied to the solution obtained by the inverse scattering transform.

We are studying the effect of higher dimensions on the interaction of DSW's. In this regard, we mention that in recent nonlinear optics experiments carried out in J. Fleischer's laboratory, interacting DSW's were observed. For example see Fig. ?? which depict various DSW interactions.

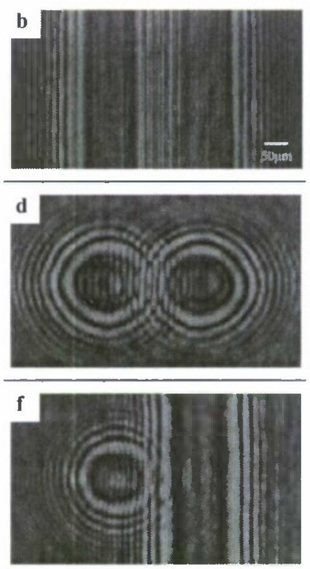

Figure 9: Interactions of DSW's; top (b): nearly one dimensional; middle (d): interacting cylindrical waves bottom (f): cylindrical DSW interacting with a one dimensional DSW.

We also analyzed another interesting problem in the theory of DSW's: the piston dispersive shock problem. Here a dispersive shock propagates with a constant speed into a dispersive fluid. The one dimensional semi-classical NLS Eq. with imposed moving boundary values is the governing equation. For small piston speed the result is a DSW separating the fluid at rest with the piston. At large enough piston speed there is a bifurcation of shock behavior and a locally periodic wave train is generated between the DSW and the piston.

Dispersive shock waves are an interesting and developing area of research which we believe will play an increasingly important role in nonlinear optics applications and other areas of physics.

PERSONNEL SUPPORTED

- Faculty: Mark J. Ablowitz
- Post-Doctoral Associates: T. Horikis
- Other (please list role) None

PUBLICATIONS

- **ACCEPTED/PUBLISHED**
- **Books/Book Chapters**

1. Nonlinear Schrödinger Equations, M.J. Ablowitz and B. Prinari *Encyclopedia of Mathematical Physics*, Eds. J-P Francoise, G.N. Naber and S.T. Tsou, Elsevier, Academic press, Amsterdam, Netherlands, 2006.
2. Solitary waves from optics to fluid dynamics, M. J. Ablowitz, and A. Docherty, Frontiers of Applied Mathematics, Eds. D-T Hsieh, M. Zhang and W. Sun, World Scientific, p. 155-186, 2007.

- **Conference Proceedings–refereed**

1. Quantum-Noise Limit on the Linewidth of Frequency Combs, B. Ilan, M. J. Ablowitz, and S. T. Cundiff, Accepted paper and presentation, Conference on Lasers and Electro-Optics (CLEO), CTuJ2, (2007).
2. Multidimensional solitons in irregular-lattice media, B. Ilan, M. J. Ablowitz, E. Schonbrun, and R. Piestun, Accepted paper and presentation, Conference on Nonlinear Photonics, Sept., (2007)

- **Journals**

1. Noise induced linewidth in frequency combs, M.J. Ablowitz, B. Ilan and S. Cundiff, *Optics Letters* **31** (2006) p. 1875-1877.

2. Solitons in two-dimensional lattices possessing defects, dislocations and quasicrystal structures, M. J. Ablowitz, B. Ilan, E. Schonbrun, and R. Piestun, *Physical Review E, Rapid Communications*, (2006) 035601.

3. Two-dimensional solitons in irregular lattice systems, M. J. Ablowitz, B. Ilan, E. Schonbrun, and R. Piestun, *Theor. Math. Phys.*, **151** (2007) 723-734.

4. On a class of nonlocal nonlinear Schrödinger equations and wave collapse, M.J. Ablowitz, I. Birkitas and B. Ilan, *Eur. Phys. J.- Spec. Top.*, **147** (2007) 343-362.

5. Interactions of dispersive shock waves, M.A. Hoefer and M.J. Ablowitz, *Physica D - Nonlin. Phenom.*, **236** (2007) 44-64.

6. Piston dispersive shock wave problem, M.A. Hoefer, M.J. Ablowitz and P. Engels, *Phys. Rev. Lett.*, **100**, (2008) 084504.

7. Solitons in dispersion-managed mode-locked lasers, M.J. Ablowitz, T.P. Horikis and B. Ilan, *Phys. Rev. A*, **77** (2008) 033814.

8. Pulse dynamics and solitons in mode-locked lasers, M.J. Ablowitz and T.P. Horikis, *Phys. Rev. A*, **78** (2008) 011802.

- **Conferences, Seminars**

- **INTERACTIONS/TRANSITIONS**

1. Conference, "Nonlinearity and randomness in complex systems", SUNY Buffalo, March 31 - April 2 2006; "Solitary waves: from optics to fluid dynamics", March 31, 2006.

2. Department of Mathematics, University of Massachusetts, "Solitary waves: from optics to fluid dynamics", April 25, 2006.

3. Department of Mechanical Engineering, University of Rochester, "What you always wanted to know about solitons but...", May 5, 2006;

4. Conference, "Frontiers in Applied Mathematics", Tsinghua University, Beijing, China, June 8-9, 2006; "Solitary waves: from optics to water waves", June 9, 2006.

5. Conference, "Nonlinear Physics Theory and Experiment", Gallipolli, Italy June 22-July 1, 2006, "Solitary waves: from optics to fluids", June 23, 2006

6. AFOSR Workshop: "Nonlinear Optics", University of Arizona, October 16-17, 2006, "Nonlinear wave propagation in irregular lattices", October 16, 2006.

7. Department of Mathematics, University of Washington, "Solitary waves: from optics to fluid dynamics", December 5, 2006

8. Departments of Mathematics: Princeton and Rutgers Universities, "M.D. Kruskal: Soltions: Discovery and Impact", February 11, 2007

9. Department of Mathematics, University of Vermont, "Nonlinear waves in optics and fluid dynamics" April 23, 2007.

10. Department of Mathematics, North Carolina State University, "Soltions: Discovery, Applications and Impact", April 25, 2007.

11. Department of Mathematics, Imperial College, London, UK, "Nonlinear waves in optics and dispersive shock waves", July 19, 2007.

12. International Conference: "WAVES 2007", July 23-27, 2007, Reading University, IMA Distinguished Lecturer, "Nonlinear waves in optics and fluid dynamics", July 23, 2007.

13. International Conference: "Partial Differential Equations", Aug. 6-9, 2007, IMPA, Rio de Janeiro, Brazil, "Dispersive shoek waves", Aug. 6, 2007.

14. AFOSR Workshop: "Nonlinear Opties", University of Arizona, September 25-26, 2007, "Pulses and dynamies in mode-loeked lasers", September 25, 2007.

15. International eonferenee: Nonlinear waves Theory and application, Beijing China. June 9-12, 2008, "Solitons and dynamies in mode-loeked lasers", June 11, 2008.

16. Department of Physics, University of Naples, Naples, Italy, "Dispersive Shock Waves", June 23, 2008.

17. AFOSR Workshop: "Nonlinear Optics", Dayton Ohio, September 10-11, 2008, "Pulses, properties and dynamies in mode-loeked lasers", September 10, 2008.

18. Conference, "International eonference on magnetics and applications", Colorado State University, September 12-14, 2008, "Pulses, properties and dynamics in mode-locked lasers", September 13, 2008.

19. Department of Physics, Colorado State University, "Nonlinear waves in optics and fluid dynamics", November 17, 2008.

- **Consultative and Advisory Functions to Other Laboratories and Agencies:** None

- **Transitions:** none

NEW DISCOVERIES, INVENTIONS, OR PATENT DISCLOSURES: None

HONORS/AWARDS: Named as one of the most highly cited people in the field of Mathematics by the ISI Web of Science, 2003-present.